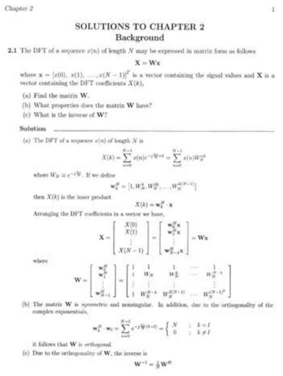

Chapter 2 1

SOLUTIONS TO CHAPTER 2
Background

2.1 The DFT of a sequence $x(n)$ of length N may be expressed in matrix form as follows

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

where $\mathbf{x} = [x(0), x(1), \ldots, x(N-1)]^T$ is a vector containing the signal values and $\mathbf{X}$ is a vector containing the DFT coefficients $X(k)$.

(a) Find the matrix $\mathbf{W}$.
(b) What properties does the matrix $\mathbf{W}$ have?
(c) What is the inverse of $\mathbf{W}$?

Solution

(a) The DFT of a sequence $x(n)$ of length N is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

where $W_N \equiv e^{-j\frac{2\pi}{N}}$. If we define

$$\mathbf{w}_k^H = [1, W_N^k, W_N^{2k}, \ldots, W_N^{k(N-1)}]$$

then $X(k)$ is the inner product

$$X(k) = \mathbf{w}_k^H \cdot \mathbf{x}$$

Arranging the DFT coefficients in a vector we have,

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{w}_0^H \mathbf{x} \\ \mathbf{w}_1^H \mathbf{x} \\ \vdots \\ \mathbf{w}_{N-1}^H \mathbf{x} \end{bmatrix} = \mathbf{W}\mathbf{x}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0^H \\ \mathbf{w}_1^H \\ \vdots \\ \mathbf{w}_{N-1}^H \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

(b) The matrix $\mathbf{W}$ is *symmetric* and nonsingular. In addition, due to the orthogonality of the complex exponentials,

$$\mathbf{w}_k^H \cdot \mathbf{w}_l = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)} = \begin{cases} N & ; \ k = l \\ 0 & ; \ k \neq l \end{cases}$$

it follows that $\mathbf{W}$ is *orthogonal*.

(c) Due to the orthogonality of $\mathbf{W}$, the inverse is

$$\mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^H$$